

Spectral decomposition of 3D Fokker - Planck differential operator

Igor A. Tanski
Moscow
povorot2@infoline.su
V&T

ABSTRACT

We construct spectral decomposition of 3D Fokker - Planck differential operator in this paper. We use the decomposition to obtain solution of Cauchy problem - and especially the fundamental solution. Then we use the decomposition to calculate macroscopic parameters of Fokker - Planck flow.

1. Fokker - Planck 3D differential operator

The Fokker - Planck equation in three dimensions is

$$\frac{\partial n}{\partial t} + \sum_{k=1}^{k=3} \left(v_k \frac{\partial n}{\partial x_k} - \alpha \frac{\partial(n v_k)}{\partial v_k} - k \frac{\partial^2 n}{\partial v_k \partial v_k} \right) = 0. \quad (1)$$

where

$n = n(t, x_k, v_k)$ - density;

t - time variable;

x_k - space coordinate;

v_k - velocity;

α - damping coefficient;

k - diffusion coefficient.

(1) is evolutionary equation with differential operator

$$L(\phi) = \sum_{k=1}^{k=3} \left(-v_k \frac{\partial \phi}{\partial x_k} + \alpha \frac{\partial(\phi v_k)}{\partial v_k} + k \frac{\partial^2 \phi}{\partial v_k \partial v_k} \right). \quad (2)$$

where $\phi = \phi(x_k, v_k)$.

We call differential operator (2) - Fokker - Planck 3D differential operator. This operator is not self-adjoint. Its adjoint operator is:

$$L^*(\psi) = \sum_{k=1}^{k=3} \left(v_k \frac{\partial \psi}{\partial x_k} - \alpha v_k \frac{\partial \psi}{\partial v_k} + k \frac{\partial^2 \psi}{\partial v_k \partial v_k} \right). \quad (3)$$

We discuss simple cyclic boundary conditions

$$\phi(x_i + a_i, v_k) = \phi(x_i, v_k), \quad -\infty < v < \infty. \quad (4)$$

where a_k - period along the corresponding coordinate axe.

2. Eigenfunctions and eigenvalues of Fokker - Planck 3D differential operator

Eigenfunctions ϕ of differential operator (2) must satisfy with differential equation

$$L(\phi) = \lambda \phi = \sum_{k=1}^{k=3} \left(-v_k \frac{\partial n}{\partial x_k} + \alpha \frac{\partial(n v_k)}{\partial v_k} + k \frac{\partial^2 n}{\partial v_k \partial v_k} \right) = 0. \quad (5)$$

and boundary conditions (4).

We search solution of equation (5) in the form

$$\phi = X_1(x_1)X_2(x_2)X_3(x_3)V_1(v_1)V_2(v_2)V_3(v_3). \quad (6)$$

or

$$\lambda + \sum_1^3 \left(v_k \frac{X_k'}{X_k} - \alpha \frac{(v_k V_k)'}{V_k} - k \frac{V_k''}{V_k} \right) = 0. \quad (7)$$

The separation of variables provides two equations

$$X_k' = c_k X_k, \quad (8)$$

$$kV_i'' + \alpha(v_i V_i)' - c_i v_i V_i = \mu_i V_i, \quad (9)$$

where c_i and μ_i are separation constants.

According to (7)

$$\lambda = \mu_1 + \mu_2 + \mu_3. \quad (10)$$

The solution of (8) and (4) is

$$X_{km} = e^{\frac{2\pi i m_k x_k}{a_k}}; \quad (11)$$

where m_k is integer and accordingly $c_k = 2\pi i \frac{m_k}{a_k}$. Now (9) takes the form

$$kV_j'' + \alpha(v_j V_j)' - 2\pi i \frac{m_j}{a_j} v_j V_j = \mu_j V_j. \quad (12)$$

We solved 1D equation (12) in our work [1]. Eigenvalues of (12) are

$$\mu_j = -\alpha n_j - k \left(\frac{2\pi m_j}{\alpha a_j} \right)^2. \quad (13)$$

Corresponding Eigenfunctions are

$$V_{m_j n_j} = \exp\left(-\frac{2\pi i m_j}{\alpha a_j} v_j\right) \exp\left(-\frac{\alpha}{2k} v_j^2\right) H_{n_j}\left(\sqrt{\frac{\alpha}{2k}}\left(v_j + \frac{4\pi i m_j k}{\alpha^2 a_j}\right)\right) \quad (14)$$

where:

- $1 \leq j \leq 3$ - coordinate number;
- $n_j \geq 0$ and m_j - integers;
- $H_n(\xi)$ - Hermite polynomial.

Hermite polynomials are defined as

$$H_n(z) = \frac{(-1)^n}{\rho} \frac{d^n}{dz^n} \rho; \quad (15)$$

where $\rho = e^{-z^2}$.

Eigenfunctions ϕ according to (6) have the form

$$\phi_{m_1 m_2 m_3 n_1 n_2 n_3} = \prod_{j=1}^{j=3} \exp\left(2\pi i \frac{m_j}{a_j} \left(x_j - \frac{v_j}{\alpha}\right)\right) \exp\left(-\frac{\alpha}{2k} v_j^2\right) H_{n_j}\left(\sqrt{\frac{\alpha}{2k}}\left(v_j + \frac{4\pi i m_j k}{\alpha^2 a_j}\right)\right). \quad (16)$$

Each eigenfunction is uniquely characterized by a set of 3 nonnegative integers n_j and a set of 3 integers m_j .

Expression for eigenfunctions of adjoint differential operator is according to [1]

$$\psi_{m_1 m_2 m_3 n_1 n_2 n_3} = \prod_{j=1}^{j=3} \exp\left(-2\pi i \frac{m_j}{a} \left(x_j + \frac{v_j}{\alpha}\right)\right) H_{n_j}\left(\sqrt{\frac{\alpha}{2k}}\left(v_j + \frac{4\pi i m_j k}{\alpha^2 a_j}\right)\right). \quad (17)$$

Evidently, these eigenfunctions are orthogonal

$$\int_0^{a_1} dx_1 \int_0^{a_2} dx_2 \int_0^{a_3} dx_3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{m_1 m_2 m_3 p_1 p_2 p_3} \psi_{n_1 n_2 n_3 q_1 q_2 q_3} dv_1 dv_2 dv_3 = 2^{p_1+p_2+p_3} p_1! p_2! p_3! \left(\frac{2\pi k}{\alpha} \right)^{\frac{3}{2}} \times \quad (18)$$

$$\times a_1 a_2 a_3 \delta_{m_1 n_1} \delta_{m_2 n_2} \delta_{m_3 n_3} \delta_{p_1 q_1} \delta_{p_2 q_2} \delta_{p_3 q_3} \exp \left[-\frac{\alpha}{2k} \left(\frac{4\pi k}{\alpha^2} \right)^2 \left(\left(\frac{m_1}{a_1} \right)^2 + \left(\frac{m_2}{a_2} \right)^2 + \left(\frac{m_3}{a_3} \right)^2 \right) \right].$$

We can write following expansion of arbitrary function $G(x_j, v_j)$

$$G(x_j, v_j) = \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} A_{m_1 m_2 m_3 p_1 p_2 p_3} \phi_{m_1 m_2 m_3 p_1 p_2 p_3}, \quad (19)$$

where coefficients $A_{m_1 m_2 m_3 p_1 p_2 p_3}$ are equal to (see (18)):

$$A_{m_1 m_2 m_3 p_1 p_2 p_3} = \frac{1}{2^{p_1+p_2+p_3} p_1! p_2! p_3!} \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \frac{1}{a_1 a_2 a_3} \exp \left[\frac{\alpha}{2k} \left(\frac{4\pi k}{\alpha^2} \right)^2 \left(\left(\frac{m_1}{a_1} \right)^2 + \left(\frac{m_2}{a_2} \right)^2 + \left(\frac{m_3}{a_3} \right)^2 \right) \right] \times \quad (20)$$

$$\times \int_0^{a_1} dx_1 \int_0^{a_2} dx_2 \int_0^{a_3} dx_3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x_j, v_j) \psi_{m_1 m_2 m_3 p_1 p_2 p_3} dv_1 dv_2 dv_3.$$

3. Solution of Cauchy problem and fundamental solution

Solution of Cauchy problem with initial condition $n_0 = G(x_j, v_j)$ is

$$n(t, x_j, v_j) = \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + k \left(\frac{2\pi}{\alpha} \right)^2 \sum_{j=1}^3 \left(\frac{m_j}{a_j} \right)^2 \right) \right] \times \quad (21)$$

$$\times A_{m_1 m_2 m_3 p_1 p_2 p_3} \phi_{m_1 m_2 m_3 p_1 p_2 p_3},$$

where coefficients $A_{m_1 m_2 m_3 p_1 p_2 p_3}$ are given by (20).

The case of

$$G(x_j, v_j) = \delta(x_1 - \xi_1) \delta(x_2 - \xi_2) \delta(x_3 - \xi_3) \delta(v_1 - u_1) \delta(v_2 - u_2) \delta(v_3 - u_3) \quad (22)$$

deserves special treatment. Here ξ_j is initial point and u_j is initial velocity.

In this case coefficients $A_{m_1 m_2 m_3 p_1 p_2 p_3}$ are equal to

$$A_{m_1 m_2 m_3 p_1 p_2 p_3} = \frac{1}{2^{p_1+p_2+p_3} p_1! p_2! p_3!} \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \frac{1}{a_1 a_2 a_3} \times \exp \left[\frac{\alpha}{2k} \left(\frac{4\pi k}{\alpha^2} \right)^2 \left(\left(\frac{m_1}{a_1} \right)^2 + \left(\frac{m_2}{a_2} \right)^2 + \left(\frac{m_3}{a_3} \right)^2 \right) \right] \psi_{m_1 m_2 m_3 p_1 p_2 p_3}(\xi_j, u_j). \quad (23)$$

This leads to following delta function expansion

$$\begin{aligned} & \delta(x_1 - \xi_1) \delta(x_2 - \xi_2) \delta(x_3 - \xi_3) \delta(v_1 - u_1) \delta(v_2 - u_2) \delta(v_3 - u_3) = \\ & = \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \frac{1}{a_1 a_2 a_3} \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} \exp \left[\frac{\alpha}{2k} \left(\frac{4\pi k}{\alpha^2} \right)^2 \left(\left(\frac{m_1}{a_1} \right)^2 + \left(\frac{m_2}{a_2} \right)^2 + \left(\frac{m_3}{a_3} \right)^2 \right) \right] \times \\ & \times \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \frac{\phi_{m_1 m_2 m_3 p_1 p_2 p_3}(x_j, v_j) \psi_{m_1 m_2 m_3 p_1 p_2 p_3}(\xi_j, u_j)}{2^{p_1+p_2+p_3} p_1! p_2! p_3!}. \end{aligned} \quad (24)$$

After substitution to (24) expressions (16) and (17), we have

$$\begin{aligned} & \delta(x_1 - \xi_1) \delta(x_2 - \xi_2) \delta(x_3 - \xi_3) \delta(v_1 - u_1) \delta(v_2 - u_2) \delta(v_3 - u_3) = \\ & = \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \frac{1}{a_1 a_2 a_3} \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \prod_{j=1}^{j=3} \exp \left(2\pi i m_j \frac{(x_j - \xi_j)}{a_j} \right) \exp \left(2\pi i m_j \frac{(v_j - u_j)}{\alpha a_j} \right) \times \\ & \times \prod_{j=1}^{j=3} e^{-\frac{\alpha}{2k} \left(v_j + \frac{4\pi i m_j k}{\alpha^2 a_j} \right)^2} \frac{1}{2^{p_j} p_j!} H_{p_j} \left(\sqrt{\frac{\alpha}{2k}} \left(v_j + \frac{4\pi i m_j k}{\alpha^2 a_j} \right) \right) H_{p_j} \left(\sqrt{\frac{\alpha}{2k}} \left(u_j + \frac{4\pi i m_j k}{\alpha^2 a_j} \right) \right). \end{aligned} \quad (25)$$

In this way we obtain finally expansion of fundamental solution of Fokker - Planck equation:

$$\begin{aligned} & G_a(t, x_i, \xi_i, v_i, u_i) = \\ & = \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \frac{1}{a_1 a_2 a_3} \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \prod_{j=1}^{j=3} e^{-kt \left(\frac{2\pi m_j}{\alpha a_j} \right)^2} \exp \left(2\pi i m_j \frac{(x_j - \xi_j)}{a_j} \right) \exp \left(2\pi i m_j \frac{(v_j - u_j)}{\alpha a_j} \right) \times \\ & \times \prod_{j=1}^{j=3} \exp(-\alpha p_j t) \frac{e^{-\frac{\alpha}{2k} \left(v_j + \frac{4\pi i m_j k}{\alpha^2 a_j} \right)^2}}{2^{p_j} p_j!} H_{p_j} \left(\sqrt{\frac{\alpha}{2k}} \left(v_j + \frac{4\pi i m_j k}{\alpha^2 a_j} \right) \right) H_{p_j} \left(\sqrt{\frac{\alpha}{2k}} \left(u_j + \frac{4\pi i m_j k}{\alpha^2 a_j} \right) \right). \end{aligned} \quad (26)$$

We can simplify (26) by explicit summation of the sum on p_j . As an instrument for this task we use following identity (see [1]).

$$\sum_{n=0}^{\infty} \frac{e^{-\frac{1}{2}(x^2+y^2)}}{2^n n!} s^n H_n(x) H_n(y) = \frac{1}{\sqrt{1-s^2}} \exp \left[\frac{x^2-y^2}{2} - \frac{(x-ys)^2}{1-s^2} \right]. \quad (27)$$

This identity is true when $|s| < 1$.

This gives after long calculations the final form of fundamental solution of initial value problem with simple cyclic boundary conditions (4):

$$G_a(t, x_i, \xi_i, v_i, u_i) = \frac{1}{a_1 a_2 a_3} \left(\frac{\alpha}{2\pi k(1-e^{-2\alpha t})} \right)^{\frac{3}{2}} \prod_{j=1}^3 \exp \left[-\frac{\alpha}{2k} \frac{(v_j - e^{-\alpha t} u_j)^2}{(1-e^{-2\alpha t})} \right] \times \quad (28)$$

$$\times \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} \prod_{j=1}^3 \exp \left[-k \left(\frac{2\pi m_j}{\alpha a_j} \right)^2 \left(t - \frac{2}{\alpha} \frac{1-e^{-\alpha t}}{1+e^{-\alpha t}} \right) \right] \exp \left[\frac{2\pi i m_j}{a_j} \left((x_j - \xi_j) + \frac{1}{\alpha} (v_j - u_j) - \frac{2}{\alpha} \frac{(v_j - e^{-\alpha t} u_j)}{(1+e^{-\alpha t})} \right) \right].$$

4. Fundamental solution for the unbounded space

Direct periods a_j to infinity and get from (28) fundamental solution for the unbounded space. The Fourier series sum is transformed to Fourier integral. Formally, we perform substitutions $\omega_j = 2\pi m_j/a_j$ and $d\omega_j \approx 2\pi/a_j$.

$$G_{\infty}(t, x_i, \xi_i, v_i, u_i) = \frac{1}{(2\pi)^3} \left(\frac{\alpha}{2\pi k(1-e^{-2\alpha t})} \right)^{\frac{3}{2}} \prod_{j=1}^3 \exp \left[-\frac{\alpha}{2k} \frac{(v_j - e^{-\alpha t} u_j)^2}{(1-e^{-2\alpha t})} \right] \times \quad (29)$$

$$\times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{j=1}^3 \exp \left[-k \left(\frac{\omega_j}{\alpha} \right)^2 \left(t - \frac{2}{\alpha} \frac{1-e^{-\alpha t}}{1+e^{-\alpha t}} \right) \right] \exp \left[i\omega_j \left((x_j - \xi_j) + \frac{1}{\alpha} (v_j - u_j) - \frac{2}{\alpha} \frac{(v_j - e^{-\alpha t} u_j)}{(1+e^{-\alpha t})} \right) \right] d\omega_1 d\omega_2 d\omega_3.$$

We perform calculations (29) using Gaussian integral

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-D\omega^2} e^{i\omega p} d\omega = \frac{1}{2\sqrt{\pi D}} e^{-\frac{p^2}{4D}}. \quad (30)$$

For our task D and p_j are equal to

$$p_j = (x_j - \xi_j) + \frac{1}{\alpha} (v_j - u_j) - \frac{2}{\alpha} \frac{(v_j - e^{-\alpha t} u_j)}{(1+e^{-\alpha t})}; \quad (31)$$

$$D = \frac{k}{\alpha^2} \left(t - \frac{2}{\alpha} \frac{1-e^{-\alpha t}}{1+e^{-\alpha t}} \right). \quad (32)$$

Use another variable D_* :

$$D_* = \frac{\alpha t (1 - e^{-2\alpha t}) - 2(1 - e^{-\alpha t})^2}{2\alpha^4}. \quad (33)$$

This variable was introduced in our former work [5] as determinant of quadratic form matrix. We have:

$$D = D_* \frac{2k\alpha}{(1 - e^{-2\alpha t})}. \quad (34)$$

Perform transformation and get another form of fundamental solution of Cauchy problem for the unbounded space:

$$\begin{aligned} G_\infty(t, x_i, \xi_i, v_i, u_i) = & \left(\frac{\alpha}{2\pi k(1 - e^{-2\alpha t})} \right)^{\frac{3}{2}} \prod_{j=1}^3 \exp \left[-\frac{\alpha}{2k} \frac{(v_j - e^{-\alpha t} u_j)^2}{(1 - e^{-2\alpha t})} \right] \times \\ & \times \frac{1}{8} \left(\frac{(1 - e^{-2\alpha t})}{2\pi k \alpha D_*} \right)^{\frac{3}{2}} \prod_{j=1}^3 \exp \left\{ -\frac{1}{4kD_*} \frac{(1 - e^{-2\alpha t})}{2\alpha} \left((x_j - \xi_j) + \frac{1}{\alpha} (v_j - u_j) - \frac{2}{\alpha} \frac{(v_j - e^{-\alpha t} u_j)^2}{(1 + e^{-\alpha t})} \right)^2 \right\}. \end{aligned} \quad (35)$$

Join two exponents

$$\begin{aligned} G_\infty(t, x_i, \xi_i, v_i, u_i) = & \left(\frac{\alpha}{2\pi k(1 - e^{-2\alpha t})} \right)^{\frac{3}{2}} \times \\ & \times \frac{1}{8} \left(\frac{(1 - e^{-2\alpha t})}{2\pi k \alpha D_*} \right)^{\frac{3}{2}} \prod_{j=1}^3 \exp \left\{ -\frac{1}{4kD_*} \left[\frac{(1 - e^{-2\alpha t})}{2\alpha} \left((x_j - \xi_j) + \frac{1}{\alpha} (v_j - u_j) - \frac{2}{\alpha} \frac{(v_j - e^{-\alpha t} u_j)^2}{(1 + e^{-\alpha t})} \right)^2 + \right. \right. \\ & \left. \left. + \frac{\alpha t (1 - e^{-2\alpha t}) - 2(1 - e^{-\alpha t})^2}{\alpha^3} \frac{(v_j - e^{-\alpha t} u_j)^2}{(1 - e^{-2\alpha t})} \right] \right\}. \end{aligned} \quad (36)$$

or

$$\begin{aligned} G_\infty(t, x_i, \xi_i, v_i, u_i) = & \frac{1}{(4\pi k \sqrt{D_*})^3} \prod_{j=1}^3 \times \\ & \times \exp \left\{ -\frac{1}{4kD_*} \left[\frac{(1 - e^{-2\alpha t})}{2\alpha} \left(\left((x_j - \xi_j) + \frac{1}{\alpha} (v_j - u_j) \right)^2 - \frac{4}{\alpha} \left((x_j - \xi_j) + \frac{1}{\alpha} (v_j - u_j) \right) \frac{(v_j - e^{-\alpha t} u_j)}{(1 + e^{-\alpha t})} \right) + \frac{t}{\alpha^2} (v_j - e^{-\alpha t} u_j)^2 \right] \right\}. \end{aligned} \quad (37)$$

In order to perform final step, we introduce another set of variables:

$$\hat{x}_i = x_i - \left(\xi_i + \frac{u_i}{\alpha} (1 - e^{-\alpha t}) \right); \quad (38)$$

$$\hat{v}_i = v_i - u_i e^{-\alpha t}.$$

These variables were introduced in [5] as integrals of characteristic equations. Subexpression of (37) is equal to

$$(x_i - \xi_i) + \frac{1}{\alpha} (v_i - u_i) = \hat{x}_i + \frac{\hat{v}_i}{\alpha}, \quad (39)$$

and the whole (37) is now

$$G_\infty(t, x_i, \xi_i, v_i, u_i) = \frac{1}{(4\pi k \sqrt{D_*})^3} \prod_{j=1}^{j=3} \exp \left\{ -\frac{1}{4kD_*} \left[\frac{(1 - e^{-2\alpha t})}{2\alpha} \left(\hat{x}_j + \frac{\hat{v}_j}{\alpha} \right)^2 - \frac{4}{\alpha} \left(\hat{x}_j + \frac{\hat{v}_j}{\alpha} \right) \frac{\hat{v}_j}{(1 + e^{-\alpha t})} \right] + \frac{t}{\alpha^2} \hat{v}_j^2 \right\}. \quad (40)$$

We see, that exponent's argument is quadratic form of new variables \hat{x}, \hat{v} .

Very simple calculation brings this expression to final form:

$$G_\infty(t, x_i, \xi_i, v_i, u_i) = \frac{1}{(4\pi k \sqrt{D_*})^3} \prod_{j=1}^{j=3} \exp \left\{ -\frac{1}{4kD_*} \left[\frac{(1 - e^{-2\alpha t})}{2\alpha} \hat{x}_j^2 - \left(\frac{2}{\alpha^2} (1 - e^{-\alpha t}) - \frac{1}{\alpha^2} (1 - e^{-2\alpha t}) \right) \hat{x}_j \hat{v}_j + \left(\frac{t}{\alpha^2} - \frac{2}{\alpha^3} (1 - e^{-\alpha t}) + \frac{1}{2\alpha^3} (1 - e^{-2\alpha t}) \right) \hat{v}_j^2 \right] \right\}. \quad (41)$$

As expected, this result coincide with obtained in [5].

We checked our considerations, comparing result with earlier one obtained by other means.

5. Calculation of macroscopic parameters

In this section we calculate averages for Eigenfunctions of Fokker Planck operator. We use the method, developed in our work [3].

First of all we get expression for Fourier transform M of eigenfunction (16) velocities. We study case of nonlimited space and therefore rewrite (16) once again, using substitutions $\omega_j = 2\pi m_j/a_j$ and $d\omega_j \approx 2\pi/a_j$.

$$\phi_{\omega_1 \omega_2 \omega_3 n_1 n_2 n_3} = \prod_{j=1}^{j=3} \exp \left(i\omega_j \left(x_j - \frac{v_j}{\alpha} \right) \right) \exp \left(-\frac{\alpha}{2k} v_j^2 \right) H_{n_j} \left(\sqrt{\frac{\alpha}{2k}} \left(v_j + \frac{2i\omega_j k}{\alpha^2} \right) \right). \quad (42)$$

The Fourier transform of eigenfunction (16) is

$$F(t, x_i, q_i) = \left(\frac{k}{2\pi\alpha} \right)^{3/2} \prod_{j=1}^{j=3} (-i)^{n_j} \left(\frac{2k}{\alpha} \right)^{n_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right)^2 \right] \left(q_j - \frac{\omega_j}{\alpha} \right)^{n_j}. \quad (43)$$

(43) is simply equation (19) from our work [1]. We only restored constant multipliers, which we previously dropped for simplicity sake. We see, that (43) is generalization of static solution from our work [3]. We get the static solution from (43), when all ω_j and n_j are zero.

We find derivatives of F easily

$$\frac{\partial F}{\partial q_j} = \left[-\frac{k}{\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right) + \frac{n_j}{\left(q_j - \frac{\omega_j}{\alpha} \right)} \right] F. \quad (44)$$

$$\frac{\partial^2 F}{\partial q_j^2} = \left[-\frac{k}{\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right) + \frac{n_j}{\left(q_j - \frac{\omega_j}{\alpha} \right)} \right]^2 F - \left[\frac{k}{\alpha} + \frac{n_j}{\left(q_j - \frac{\omega_j}{\alpha} \right)^2} \right] F. \quad (45)$$

$$\frac{\partial^2 F}{\partial q_i \partial q_j} = \left[-\frac{k}{\alpha} \left(q_i + \frac{\omega_i}{\alpha} \right) + \frac{n_i}{\left(q_i - \frac{\omega_i}{\alpha} \right)} \right] \left[-\frac{k}{\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right) + \frac{n_j}{\left(q_j - \frac{\omega_j}{\alpha} \right)} \right] F. \quad (46)$$

when $i \neq j$.

We can write (45) and (46) as a single expression, using Kronecker delta

$$\frac{\partial^2 F}{\partial q_i \partial q_j} = \left[-\frac{k}{\alpha} \left(q_i + \frac{\omega_i}{\alpha} \right) + \frac{n_i}{\left(q_i - \frac{\omega_i}{\alpha} \right)} \right] \left[-\frac{k}{\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right) + \frac{n_j}{\left(q_j - \frac{\omega_j}{\alpha} \right)} \right] F - \delta_{ij} \left[\frac{k}{\alpha} + \frac{n_j}{\left(q_j - \frac{\omega_j}{\alpha} \right)^2} \right] F. \quad (47)$$

Now let us return to expansion (21) for solution of Cauchy problem. We use expression (43) for Fourier transformation of each term in (21) and get in this way expression for Fourier transform of arbitrary solution

$$M(t, x_j, q_j) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \quad (48)$$

$$\times A_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3} \left(\frac{k}{2\pi\alpha} \right)^{3/2} \prod_{j=1}^{j=3} (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right)^2 \right] \left(q_j - \frac{\omega_j}{\alpha} \right)^{p_j} d\omega_1 d\omega_2 d\omega_3.$$

$$\begin{aligned}
 \frac{\partial M}{\partial q_k} &= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \\
 &\quad \times A_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3} \left(\frac{k}{2\pi\alpha} \right)^{3/2} \left[-\frac{k}{\alpha} \left(q_k + \frac{\omega_k}{\alpha} \right) + \frac{p_k}{\left(q_k - \frac{\omega_k}{\alpha} \right)} \right] \times \\
 &\quad \times \prod_{j=1}^{j=3} (-i)^{n_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right)^2 \right] \left(q_j - \frac{\omega_j}{\alpha} \right)^{p_j} d\omega_1 d\omega_2 d\omega_3.
 \end{aligned} \tag{49}$$

$$\begin{aligned}
 \frac{\partial^2 M}{\partial q_k^2} &= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \\
 &\quad \times A_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3} \left(\frac{k}{2\pi\alpha} \right)^{3/2} \left[\left[-\frac{k}{\alpha} \left(q_k + \frac{\omega_k}{\alpha} \right) + \frac{p_k}{\left(q_k - \frac{\omega_k}{\alpha} \right)} \right]^2 - \left[\frac{k}{\alpha} + \frac{p_k}{\left(q_k - \frac{\omega_k}{\alpha} \right)^2} \right] \right] \times \\
 &\quad \times \prod_{j=1}^{j=3} (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right)^2 \right] \left(q_j - \frac{\omega_j}{\alpha} \right)^{p_j} d\omega_1 d\omega_2 d\omega_3.
 \end{aligned} \tag{50}$$

$$\begin{aligned}
 \frac{\partial^2 M}{\partial q_k \partial q_l} &= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \\
 &\quad \times A_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3} \left(\frac{k}{2\pi\alpha} \right)^{3/2} \left[\left[-\frac{k}{\alpha} \left(q_k + \frac{\omega_k}{\alpha} \right) + \frac{p_k}{\left(q_k - \frac{\omega_k}{\alpha} \right)} \right] \left[-\frac{k}{\alpha} \left(q_l + \frac{\omega_l}{\alpha} \right) + \frac{p_l}{\left(q_l - \frac{\omega_l}{\alpha} \right)} \right] - \delta_{kl} \left[\frac{k}{\alpha} + \frac{p_l}{\left(q_l - \frac{\omega_l}{\alpha} \right)^2} \right] \right] \times \\
 &\quad \times \prod_{j=1}^{j=3} (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right)^2 \right] \left(q_j - \frac{\omega_j}{\alpha} \right)^{p_j} d\omega_1 d\omega_2 d\omega_3
 \end{aligned} \tag{51}$$

when $k \neq l$.

Density is equal to

$$\rho = (2\pi)^3 M_0 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \tag{52}$$

$$\times A_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3} \left(\frac{k}{2\pi\alpha} \right)^{3/2} \prod_{j=1}^{j=3} (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(\frac{\omega_j}{\alpha} \right)^2 \right] \left(-\frac{\omega_j}{\alpha} \right)^{p_j} d\omega_1 d\omega_2 d\omega_3$$

Average velocities are

$$\begin{aligned} u_k = (2\pi)^3 \frac{i}{\rho} M_k &= \frac{i}{\rho} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \\ &\times A_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3} \left(\frac{k}{2\pi\alpha} \right)^{3/2} \left[-\frac{k}{\alpha} \frac{\omega_k}{\alpha} - \frac{\alpha p_k}{\omega_k} \right] \times \\ &\times \prod_{j=1}^{j=3} (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(\frac{\omega_j}{\alpha} \right)^2 \right] \left(-\frac{\omega_j}{\alpha} \right)^{p_j} d\omega_1 d\omega_2 d\omega_3. \end{aligned} \quad (53)$$

The current of momentum tensor is

$$\begin{aligned} J_{kl} = -(2\pi)^3 M_{kl} &= - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \\ &\times A_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3} \left(\frac{k}{2\pi\alpha} \right)^{3/2} \left[\left[-\frac{k}{\alpha} \frac{\omega_k}{\alpha} - \frac{\alpha p_k}{\omega_k} \right] \left[-\frac{k}{\alpha} \frac{\omega_l}{\alpha} - \frac{\alpha p_l}{\omega_l} \right] - \delta_{kl} \left[\frac{k}{\alpha} + \frac{\alpha^2 p_l}{\omega_l^2} \right] \right] \times \\ &\times \prod_{j=1}^{j=3} (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(\frac{\omega_j}{\alpha} \right)^2 \right] \left(-\frac{\omega_j}{\alpha} \right)^{p_j} d\omega_1 d\omega_2 d\omega_3. \end{aligned} \quad (54)$$

Stresses are equal to

$$\sigma_{kl} = \rho u_k u_l - J_{kl} = (2\pi)^3 \left(M_{kl} - \frac{M_k M_l}{M_0} \right). \quad (55)$$

Hydrostatic pressure is

$$p = -\frac{1}{3} \sum_{k=1}^3 \sigma_{kk} = -\frac{1}{3} (2\pi)^3 \sum_{k=1}^3 \left(M_{kk} - \frac{M_k M_k}{M_0} \right). \quad (56)$$

6. Calculation of macroscopic parameters of fundamental solution

Coefficients $A_{\omega_1\omega_2\omega_3p_1p_2p_3}$ for the case of fundamental solution for unbounded space according to (23) and substitutions $\omega_j = 2\pi m_j/a_j$ and $d\omega_j \approx 2\pi/a_j$ are equal to

$$A_{\omega_1\omega_2\omega_3p_1p_2p_3} = \frac{1}{2^{p_1+p_2+p_3} p_1! p_2! p_3!} \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \frac{1}{a_1 a_2 a_3} \times \exp \left[\frac{2k}{\alpha^3} \left(\omega_1^2 + \omega_2^2 + \omega_3^2 \right) \right] \psi_{\omega_1\omega_2\omega_3p_1p_2p_3}(\xi_m, u_m). \quad (57)$$

Expression for M is (see (48))

$$\begin{aligned} M(t, x_j, q_j) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \\ &\times \frac{1}{2^{p_1+p_2+p_3} p_1! p_2! p_3!} \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \exp \left[\frac{2k}{\alpha^3} \left(\omega_1^2 + \omega_2^2 + \omega_3^2 \right) \right] \times \\ &\times \left(\frac{k}{2\pi\alpha} \right)^{3/2} \prod_{j=1}^3 (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right)^2 \right] \left(q_j - \frac{\omega_j}{\alpha} \right)^{p_j} \psi_{\omega_1\omega_2\omega_3p_1p_2p_3}(\xi_m, u_m) d\omega_1 d\omega_2 d\omega_3 = \\ &= \frac{1}{(2\pi)^3} \left(\frac{\alpha^3}{2\pi k \theta} \right)^{3/2} \prod_{j=1}^3 \exp(-ie^{-\alpha t} v_{0j} q_j) \exp \left[-\frac{1}{\theta} \left(\frac{\alpha^3}{2k} \bar{x}_j \bar{x}_j + i\alpha(1-e^{-\alpha t})^2 \bar{x}_j q_j + \frac{k}{\alpha} \left(\alpha t (1-e^{-2\alpha t}) - 2(1-e^{-\alpha t})^2 \right) q_j q_j \right) \right]. \end{aligned} \quad (58)$$

The last expression we know from [3]. In this expression we use following definitions:

$$\theta = 2\alpha t - (1 - e^{-\alpha t}) (3 - e^{-\alpha t}), \quad (59)$$

and \hat{x}_i - as in (38).

The first expression in (58) looks rather awkward, but in fact each term in it is a product of three factors, each dependent only on one of p_j .

Therefore we can write (58) as a product of three independent sums

However, this decomposition not possible in the common case. We could not expect, that we can simplify expressions (48), (52-56) in the common case.

To prove (58) we use following identity (ref. [4])

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = \exp(2xz - z^2). \quad (60)$$

In our case this give

$$\begin{aligned} \sum_{p=0}^{p=\infty} \frac{\exp(-\alpha p t)}{2^p p!} (-i)^p \left(\frac{2k}{\alpha}\right)^{p/2} \left(q - \frac{\omega}{\alpha}\right)^p H_p\left(\sqrt{\frac{\alpha}{2k}} \left(u + \frac{2i\omega k}{\alpha^2}\right)\right) = \\ = \exp\left[-ie^{-\alpha t} \left(u + \frac{2i\omega k}{\alpha^2}\right) \left(q - \frac{\omega}{\alpha}\right)\right] \exp\left[e^{-2\alpha t} \frac{k}{2\alpha} \left(q - \frac{\omega}{\alpha}\right)^2\right]. \end{aligned} \quad (61)$$

This provide instrument to calculate series in (58). The result is :

$$\begin{aligned} M(t, x_j, \xi_j, q_j, u_j) = \frac{1}{(2\pi)^6} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{j=1}^{j=3} \exp\left[-k \left(\frac{\omega_j}{\alpha}\right)^2 t\right] \exp\left(i\omega_j(x_j - \xi_j - \frac{u_j}{\alpha})\right) \times \\ \times \exp\left[\frac{\alpha}{2k} \left(\frac{2\omega_j k}{\alpha^2}\right)^2\right] \exp\left[-\frac{k}{2\alpha} \left(q_j + \frac{\omega_j}{\alpha}\right)^2\right] \exp\left[-ie^{-\alpha t} \left(u_j + \frac{2i\omega k}{\alpha^2}\right) \left(q_j - \frac{\omega_j}{\alpha}\right)\right] \exp\left[e^{-2\alpha t} \frac{k}{2\alpha} \left(q_j - \frac{\omega_j}{\alpha}\right)^2\right] d\omega_1 d\omega_2 d\omega_3. \end{aligned} \quad (62)$$

We collect terms with equal powers of ω_j :

$$\begin{aligned} M(t, x_j, \xi_j, q_j, u_j) = \frac{1}{(2\pi)^6} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{j=1}^{j=3} \exp\left[-\omega_j^2 \left(\frac{kt}{\alpha^2} - \frac{\alpha}{2k} \frac{4k^2}{\alpha^4} + \frac{k}{2\alpha} \frac{1}{\alpha^2} + e^{-\alpha t} \frac{2k}{\alpha^3} - e^{-2\alpha t} \frac{k}{2\alpha^3}\right)\right] \times \\ \times \exp\left[\omega_j \left(i(x_j - \xi_j - \frac{u_j}{\alpha}) - \frac{k}{2\alpha} \frac{2q_j}{\alpha} - ie^{-\alpha t} \left(-\frac{u_j}{\alpha} + \frac{2ik}{\alpha^2} q_j\right) - e^{-2\alpha t} \frac{k}{2\alpha} \frac{2q_j}{\alpha} \frac{1}{\alpha}\right)\right] \times \\ \times \exp\left[\left(-\frac{k}{2\alpha} q_j^2 - ie^{-\alpha t} u_j q_j + e^{-2\alpha t} \frac{k}{2\alpha} q_j^2\right)\right] d\omega_1 d\omega_2 d\omega_3. \end{aligned} \quad (63)$$

or

$$\begin{aligned} M(t, x_j, \xi_j, q_j, u_j) = \frac{1}{(2\pi)^6} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{j=1}^{j=3} \exp\left[-\omega_j^2 \left(\frac{kt}{\alpha^2} - \frac{3}{2} \frac{k}{\alpha^3} + e^{-\alpha t} \frac{2k}{\alpha^3} - e^{-2\alpha t} \frac{k}{2\alpha^3}\right)\right] \times \\ \times \exp\left[\omega_j \left(i(x_j - \xi_j - (1 - e^{-\alpha t}) \frac{u_j}{\alpha}) - \frac{kq_j}{\alpha^2} (1 - e^{-\alpha t})^2\right)\right] \times \\ \times \exp\left[-\frac{k}{2\alpha} q_j^2 (1 - e^{-2\alpha t}) - ie^{-\alpha t} u_j q_j\right] d\omega_1 d\omega_2 d\omega_3. \end{aligned} \quad (64)$$

We use following identity to calculate integrals in (64)

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-D\omega^2} e^{i\omega f} d\omega = \frac{1}{2\sqrt{\pi D}} e^{-\frac{f^2}{4D}}. \quad (65)$$

where detailed expressions are

$$D = \frac{kt}{\alpha^2} - \frac{3}{2} \frac{k}{\alpha^3} + e^{-\alpha t} \frac{2k}{\alpha^3} - e^{-2\alpha t} \frac{k}{2\alpha^3} = \frac{k}{\alpha^3} \theta. \quad (66)$$

$$f_j = (x_j - \xi_j - (1 - e^{-\alpha t}) \frac{u_j}{\alpha}) + i \frac{kq_j}{\alpha^2} (1 - e^{-\alpha t})^2. \quad (67)$$

The result of our calculations is

$$M(t, x_j, \xi_j, q_j, u_j) = \frac{1}{(2\pi)^3} \frac{1}{8(\pi D)^{3/2}} \prod_{j=1}^{j=3} \exp \left\{ -\frac{1}{4D} \left[(x_j - \xi_j - (1 - e^{-\alpha t}) \frac{u_j}{\alpha}) + \frac{ikq_j}{\alpha^2} (1 - e^{-\alpha t})^2 \right]^2 \right\} \times \exp \left[-\frac{k}{2\alpha} q_j^2 (1 - e^{-2\alpha t}) - ie^{-\alpha t} u_j q_j \right]. \quad (68)$$

Return to "hated" variables (recall (38)):

$$M(t, x_j, \xi_j, q_j, u_j) = \frac{1}{(2\pi)^3} \left(\frac{\alpha^3}{2\pi k \theta} \right)^{3/2} \prod_{j=1}^{j=3} \exp \left(-ie^{-\alpha t} u_j q_j \right) \times \exp \left\{ -\frac{\alpha^3}{2k\theta} \left[\hat{x} + \frac{ikq_j}{\alpha^2} (1 - e^{-\alpha t})^2 \right]^2 - \frac{k}{2\alpha} q_j^2 (1 - e^{-2\alpha t}) \right\}. \quad (69)$$

We get following expression, which is identical to last expression of (58).

$$M(t, x_j, \xi_j, q_j, u_j) = \frac{1}{(2\pi)^3} \left(\frac{\alpha^3}{2\pi k \theta} \right)^{3/2} \prod_{j=1}^{j=3} \exp \left(-ie^{-\alpha t} u_j q_j \right) \times \exp \left\{ -\frac{1}{\theta} \left[\frac{\alpha^3}{2k} \hat{x}_j + i\alpha(1 - e^{-\alpha t})^2 \hat{x}_j q_j + \frac{k}{2\alpha} \left(2\alpha t(1 - e^{-2\alpha t}) - 4(1 - e^{-\alpha t})^2 \right) q_j^2 \right] \right\} = \frac{1}{(2\pi)^3} \left(\frac{\alpha^3}{2\pi k \theta} \right)^{1/2} \exp(-ie^{-\alpha t} u_j q_j) \exp \left[-\frac{1}{\theta} \left(\frac{\alpha^3}{2k} \hat{x}_j \hat{x}_j + i\alpha(1 - e^{-\alpha t})^2 \hat{x}_j q_j + \frac{k}{\alpha} \left(\alpha t (1 - e^{-2\alpha t}) - 2(1 - e^{-\alpha t})^2 \right) q_j^2 \right) \right]. \quad (70)$$

In the following we spare such long calculations and provide only their results.

The density is

$$\begin{aligned}
 \rho &= (2\pi)^3 M_0 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \\
 &\times \frac{1}{2^{p_1+p_2+p_3} p_1! p_2! p_3!} \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \exp \left[\frac{2k}{\alpha^3} (\omega_1^2 + \omega_2^2 + \omega_3^2) \right] \times \\
 &\times \left(\frac{k}{2\pi\alpha} \right)^{3/2} \prod_{j=1}^3 (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(\frac{\omega_j}{\alpha} \right)^2 \right] \left(-\frac{\omega_j}{\alpha} \right)^{p_j} \psi_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3}(\xi_m, u_m) d\omega_1 d\omega_2 d\omega_3 = \\
 &= \left(\frac{\alpha^3}{2\pi k \theta} \right)^{3/2} \exp \left(-\frac{1}{\theta} \frac{\alpha^3}{2k} \hat{x}_k \hat{x}_k \right).
 \end{aligned} \tag{71}$$

Average velocities are

$$\begin{aligned}
 u_k &= (2\pi)^3 \frac{i}{\rho} M_k = \frac{i}{\rho} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \\
 &\times \frac{1}{2^{p_1+p_2+p_3} p_1! p_2! p_3!} \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \exp \left[\frac{2k}{\alpha^3} (\omega_1^2 + \omega_2^2 + \omega_3^2) \right] \left(\frac{k}{2\pi\alpha} \right)^{3/2} \left[-\frac{k}{\alpha} \frac{\omega_k}{\alpha} - \frac{\alpha p_k}{\omega_k} \right] \times \\
 &\times \prod_{j=1}^{j=3} (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(\frac{\omega_j}{\alpha} \right)^2 \right] \left(-\frac{\omega_j}{\alpha} \right)^{p_j} \psi_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3}(\xi_m, u_m) d\omega_1 d\omega_2 d\omega_3 = \\
 &= v_{0i} e^{-\alpha t} + \alpha (1 - e^{-\alpha t})^2 \frac{\hat{x}_i}{\theta}.
 \end{aligned} \tag{72}$$

The current of momentum tensor is

$$\begin{aligned}
 J_{kl} &= -(2\pi)^3 M_{kl} = - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \exp \left[-t \left(\alpha \sum_{j=1}^3 p_j + \frac{k}{\alpha^2} \sum_{j=1}^3 \omega_j^2 \right) \right] \times \\
 &\times \frac{1}{2^{p_1+p_2+p_3} p_1! p_2! p_3!} \left(\frac{\alpha}{2\pi k} \right)^{\frac{3}{2}} \exp \left[\frac{2k}{\alpha^3} (\omega_1^2 + \omega_2^2 + \omega_3^2) \right] \left(\frac{k}{2\pi\alpha} \right)^{3/2} \left[\left[-\frac{k}{\alpha} \frac{\omega_k}{\alpha} - \frac{\alpha p_k}{\omega_k} \right] \left[-\frac{k}{\alpha} \frac{\omega_l}{\alpha} - \frac{\alpha p_l}{\omega_l} \right] - \delta_{kl} \left[\frac{k}{\alpha} + \frac{\alpha^2 p_l}{\omega_l^2} \right] \right] \times \\
 &\times \prod_{j=1}^{j=3} (-i)^{p_j} \left(\frac{2k}{\alpha} \right)^{p_j/2} \exp(i\omega_j x_j) \exp \left[-\frac{k}{2\alpha} \left(q_j + \frac{\omega_j}{\alpha} \right)^2 \right] \left(q_j - \frac{\omega_j}{\alpha} \right)^{p_j} \psi_{\omega_1 \omega_2 \omega_3 p_1 p_2 p_3}(\xi_m, u_m) d\omega_1 d\omega_2 d\omega_3 = \\
 &= \left(\frac{\alpha^3}{2\pi k \theta} \right)^{3/2} \exp \left(-\frac{1}{\theta} \frac{\alpha^3}{2k} \hat{x}_k \hat{x}_k \right) \left[\left(v_{0k} e^{-\alpha t} + \alpha (1 - e^{-\alpha t})^2 \frac{\hat{x}_k}{\theta} \right) \left(v_{0l} e^{-\alpha t} + \alpha (1 - e^{-\alpha t})^2 \frac{\hat{x}_l}{\theta} \right) + \delta_{kl} \frac{2k}{\alpha \theta} \left(\alpha t (1 - e^{-2\alpha t}) - 2 (1 - e^{-\alpha t})^2 \right) \right].
 \end{aligned} \tag{73}$$

APPENDIX

We obtained expressions for eigenfunctions (16), (17), and expression for fundamental solution (26) for the case of cyclic boundary conditions (4). In this appendix we give formulae for the case of unbounded space. They are:

$$\phi_{\omega_1\omega_2\omega_3p_1p_2p_3} = \prod_{j=1}^{j=3} \exp\left(i\omega_j\left(x_j - \frac{v_j}{\alpha}\right)\right) \exp\left(-\frac{\alpha}{2k} v_j^2\right) H_{p_j}\left(\sqrt{\frac{\alpha}{2k}}\left(v_j + \frac{2i\omega_j k}{\alpha^2}\right)\right). \quad (\text{A1})$$

$$\psi_{\omega_1\omega_2\omega_3p_1p_2p_3} = \prod_{j=1}^{j=3} \exp\left(-i\omega_j\left(x_j + \frac{v_j}{\alpha}\right)\right) H_{p_j}\left(\sqrt{\frac{\alpha}{2k}}\left(v_j + \frac{2i\omega_j k}{\alpha^2}\right)\right). \quad (\text{A2})$$

$$G(t, x_i, \xi_i, v_i, u_i) = \quad (\text{A3})$$

$$\begin{aligned} &= \left(\frac{\alpha}{2\pi k}\right)^{\frac{3}{2}} \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \prod_{j=1}^{j=3} e^{-kt\left(\frac{\omega_j}{\alpha}\right)^2} \exp\left(i\omega_j(x_j - \xi_j)\right) \exp\left(i\omega_j \frac{(v_j - u_j)}{\alpha}\right) \times \\ &\times \prod_{j=1}^{j=3} \exp(-\alpha p_j t) \frac{e^{-\frac{\alpha}{2k}\left(v_j + \frac{2i\omega_j k}{\alpha^2}\right)^2}}{2^{p_j} p_j!} H_{p_j}\left(\sqrt{\frac{\alpha}{2k}}\left(v_j + \frac{2i\omega_j k}{\alpha^2}\right)\right) H_{p_j}\left(\sqrt{\frac{\alpha}{2k}}\left(u_j + \frac{2i\omega_j k}{\alpha^2}\right)\right) d\omega_1 d\omega_2 d\omega_3. \end{aligned}$$

$$\begin{aligned} A_{\omega_1\omega_2\omega_3p_1p_2p_3} &= \frac{1}{2^{p_1+p_2+p_3} p_1! p_2! p_3!} \left(\frac{\alpha}{2\pi k}\right)^{\frac{3}{2}} \frac{1}{(2\pi)^3} \exp\left[\frac{2k}{\alpha^3} (\omega_1^2 + \omega_2^2 + \omega_3^2)\right] \times \\ &\times \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x_j, v_j) \psi_{\omega_1\omega_2\omega_3p_1p_2p_3} dv_1 dv_2 dv_3. \end{aligned} \quad (\text{A4})$$

DISCUSSION

In this paper we get the closed form expression for eigenfunctions and eigenvalues of 3D Fokker - Planck differential operator. This expression is known from previous works - see [5]. We use the decomposition to obtain solution of Cauchy problem for 3D Fokker - Planck equation. Both periodic problem with cyclic boundary conditions and the unbounded space case are considered. We calculate macroscopic parameters of 3D Fokker - Planck flow also.

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